18.100A Practice problems for Chapter 1-22,24,25

The final exam will take place on May 22nd, Tuesday 1:30-4:30.

As an open book exam, during the exam you can see
1. the textbook : *Introduction to Real Analysis* by A. Mattuck,
2. notes, copies, and scratch papers (at most 500 sheets of paper).

However, the following are NOT allowed to use
1. electronic devices.
2. the other books except the textbook.

When you write the poofs of problems, you can cite Theorems, Properties, and examples with proofs in the textbook Chapter 1-22,24,25. Moreover, a sheet of facts will be given and you can cite them.

However, you can not use exercises and problems in the textbook as well as problem sets, practice problems, and their solutions. If you have copies of the solutions and want to use them, please rewrite the proofs.

Review: 5 in pset 4, 2-10 in pset 5, 5-11 in pset 6, 1-5 in pset 7.

Problem 1. *Determine whether the following statements are true or false. If false then provide a counterexample. You don't need to verify why it is a counterexample.*

(1) *Suppose* $f(x)$ *is continuous on an interval* I*. Then,* $f(x)$ *is bounded on* I*.*
(2) *Suppose* $\lim\limits_{x\to 0^+} f(x) = +\infty$*. Then,* $f(x)$ *is not continuous on* $(0,1)$*.*
(3) *Suppose* $f(x)$ *is continuous on an interval* I*. Then,* $f(x)$ *is uniformly continuous on* I*.*
(4) *Suppose* $f(x)$ *is continuous on* $[0,1]$ *and* $f(0) < 0$*,* $f(1) > 0$*. Then,* $f(x)$ *has a unique zero in* $[0,1]$*.*
(5) *Suppose* $f(x)$ *is continuous and bounded on* $[0,+\infty)$*. Then,* $f(x)$ *has the maximum on* $[0,+\infty)$*.*
(6) *Suppose* $f(x)$ *is infinitely many times differentiable at* 0*. Then, the Taylor series centered at* 0 *converges to* $f(x)$ *in a neighorhood of* 0*.*
(7) *Suppose* $f(x)$ *is a polynomial. Then,* $f(x)$ *is the same to its Taylor series.*
(8) *Suppose* $\int_{0^+}^{1^-} f(x)dx$ *converges. Then,* $\int_{0^+}^{1^-} f^2(x)dx$ *converges.*
(9) *Suppose* $f(x)$ *and* $g(x)$ *are integrable on* $I = [a,b]$*. Then,* $f(x)g(x)$ *is integrable on* I*.*
(10) *Suppose each* $f_n(x)$ *is bounded on* $I = [a,b]$ *and* $f_n(x)$ *converges to* $f(x)$*. Then,* $f(x)$ *is bounded on* I*.*
(11) *Let* I_n *are open intervals. Then,* $\cap_{n=1}^{\infty} I_n$ *is the empty set or an open interval.*
(12) *An open set* U *is not a union of closed sets.*

Proof for (1). F: Counterexample $f(x) = x$ and $I = \mathbb{R}$. $\square$

Proof for (2). F: Counterexample $f(x) = 1/x$. $\square$

Proof for (3). F: Counterexample $f(x) = x^2$ and $I = \mathbb{R}$. $\square$

Proof for (4). F: Counterexample $f(x) = (x - \frac{1}{2})(x - \frac{1}{3})(x - \frac{1}{4})$. $\square$

Proof for (5). F: Counterexample $f(x) = 1 - \frac{1}{x+1}$. $\square$

Proof for (6). F: Counterexample $f(x) = e^{-\frac{1}{x}}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. $\square$

Proof for (7). T $\square$

Proof for (8). F: Counterexample $f(x) = \frac{1}{\sqrt{x}}$. $\square$

Proof for (9). T $\square$

Proof for (10). F: Counterexample, $a = 0$, $b = 1$, $f_n(0) = 0$, $f_n(x) = n$ for $0 < x < \frac{1}{n}$, and $f_n(x) = \frac{1}{x}$ for $x \geq \frac{1}{n}$. Then, each f_n is bounded because we have$0 \leq f_n(x) \leq n$. However, $\lim_{n\to\infty} f_n(x) = f(x)$ where $f(0) = 0$ and $f(x) = 1/x$ for $x > 0$. $\square$

Proof for (11). F: Counterexample $I_n = (-\frac{1}{n}, \frac{1}{n})$. Then, $\cap_{n=1}^{\infty} I_n = \{0\}$ is a compact set. $\square$

Proof for (12). F: Counterexample $(-1, 1) = \cup_{|x|<1}\{x\}$. Namely, $(-1, 1)$ is an open interval. However, $\{x\}$ are closed sets. $\square$

1. Taylor series

Problem 2. *Find the third order Taylor polynomial $T_3(x)$ of $f(x) = e^x \sin x$ at 0, and show that $|f(x) - T_3(x)| < .02$ for $|x| < .5$.*

(Fact : $\sqrt{e} < 1.75$.)

Proof. Compute

$$f' = e^x(\sin x + \cos x), \qquad f'' = 2e^x \cos x,$$
$$f^{(3)} = 2e^x(-\sin x + \cos x), \qquad f^{(4)} = -4e^x \sin x.$$

Therefore, if $|c| < .5$ then $|f^{(4)}(c)| = 4e^c|\sin c| \leq 4e^c < 7$. By the remainder theorem, for each x there exists $|c| \leq |x| < .5$ such that

$$|f(x) - T_3(x)| = |R_3(x)| = \frac{|f^{(4)}(c)|}{4!}|c|^4 < \frac{7}{24 \cdot 16} < \frac{1}{50}.$$

$\square$

Problem 3. *Find the Taylor series of $f(x) = x^3 + 2x^2 - 7$ at 1.*

Proof. Compute

$$f'(x) = 3x^2 + 4x, \qquad f''(x) = 6x + 4, \qquad f^{(3)}(x) = 6, \qquad f^{(4)}(x) = 0.$$

Hence,

$$f(1) = -4, \qquad f'(1) = 7, \qquad f''(1) = 10, \qquad f^{(3)}(1) = 6, \qquad f^{(n)}(1) = 0,$$

for $n \geq 4$. Thus, Taylor series at 1 is

$$T(x) = -4 + 7(x-1) + 5(x-1)^2 + (x-1)^3.$$

□

2. Continuity

Prove the following statements in this section.

Problem 4. *$f(x, y)$ is defined by $f(x, y) = \sqrt{x^2 + y^2}\cos(\frac{1}{x^2+y^2})$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Then, $f(x, y)$ is continuous at $(0, 0)$.*

Proof. Given $\epsilon > 0$, if $\|(x, y)\| < \epsilon$ and $(x, y) \neq (0, 0)$ then

$$|f(x, y) - f(0, 0)| = \sqrt{x^2 + y^2}|\cos(x^2 + y^2)^{-1}| \leq \sqrt{x^2 + y^2} = \|(x, y)\| < \epsilon,$$

namely

$$\lim_{\|(x,y)\| \to 0} f(x, y) = f(0, 0).$$

Hence, the theorem 24.5A gives the desired result. □

Problem 5. *$f(x, y)$ is defined by $f(x, y) = xy(x^2 + y^2)^{-\frac{2}{3}}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Then, $f(x, y)$ is continuous at $(0, 0)$.*

Hint: $2xy \leq x^2 + y^2$.

Proof. Given $\epsilon > 0$, if $\|(x, y)\| < (2\epsilon)^{\frac{3}{2}}$ and $(x, y) \neq (0, 0)$ then

$$\begin{aligned} |f(x, y) - f(0, 0)| = \frac{|xy|}{(x^2 + y^2)^{\frac{2}{3}}} &\leq \frac{x^2 + y^2}{2(x^2 + y^2)^{\frac{2}{3}}} \\ &= \frac{1}{2}(x^2 + y^2)^{\frac{1}{3}} = \frac{1}{2}\|(x, y)\|^{\frac{2}{3}} < \epsilon \end{aligned}$$

namely

$$\lim_{\|(x,y)\| \to 0} f(x, y) = f(0, 0).$$

Hence, the theorem 24.5A gives the desired result. □

Problem 6. *$f(x, y)$ is defined by $f(x, y) = x \sin\big(\frac{1}{x^2 + y^2}\big)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Then, $f(x, y)$ is continuous at $(0, 0)$.*

Proof. Given $\epsilon > 0$, if $\|(x,y)\| < \epsilon$ and $(x,y) \neq (0,0)$ then

$$|f(x,y) - f(0,0)| = |x||\sin(x^2+y^2)^{-1}| \leq |x| \leq \sqrt{x^2+y^2} = \|(x,y)\| < \epsilon,$$

namely

$$\lim_{\|(x,y)\|\to 0} f(x,y) = f(0,0).$$

Hence, the theorem 24.5A gives the desired result. □

Problem 7. *$f(x,y)$ is defined by $f(x,y) = \sqrt{x^2+y^2}\cos\big(1/y\big)$ for $y \neq 0$ and $f(x,0) = 0$. Then, $f(x,y)$ is not continuous.*

Proof. We have

$$f(1, \frac{1}{2n\pi}) = \sqrt{1+(2n\pi)^{-2}}\cos(2n\pi) = \sqrt{1+(2n\pi)^{-2}} \to 1,$$

as $n \to +\infty$. Since $\lim_{n\to\infty} \frac{1}{2n\pi} = 0$, if f is continuous at $(1,0)$ then we have

$$0 = f(1,0) = \lim_{n\to\infty} f(1, \frac{1}{2n\pi}) = 1,$$

by the theorem 24.5A. Contradiction. □

Problem 8. *$f(x) = x\sin(1/x)$ is uniformly continuous on $(0,+\infty)$.*

Proof. We define $g(x) = x\sin(1/x)$ for $x > 0$ and $g(0) = 0$. Then, given $\epsilon > 0$ if $x \in [0,\epsilon)$

$$|g(x) - g(0)| = |x||\sin(1/x)| \leq |x| < \epsilon,$$

namely $g(x)$ is continuous at 0. Moreover, $g(x) = x\sin(1/x)$ is continuous at $x > 0$ by the theorem 11.4C, example 11.1B, and theorem 11.4D. Therefore, $g(x)$ is continuous on the compact interval $[0,2]$. So, it is uniformly continuous on $[0,2]$ by the theorem 13.5. Hence, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|g(x) - g(y)| < \epsilon$ if $|x-y| < \delta_1$ and $x, y \in [0,2]$.

Next, for $x \geq 1$ we have

$$\begin{aligned}|g'(x)| &= |\sin(1/x) - x^{-1}\cos(1/x)| \\ &\leq |\sin(1/x)| + |x|^{-1}|\cos(1/x)| \leq 1 + |x|^{-1} \leq 2.\end{aligned}$$

Hence, the fundamental theorem of calculus shows

$$|g(x) - g(x+h)| = |\int_x^{x+h} g'(t)dt| \leq \int_x^{x+h} |g'(t)|dt \leq \int_x^{x+h} 2dt = 2h,$$

for $x \geq 1$ and $h > 0$. Thus, given $\epsilon > 0$ if $|x-y| < \epsilon/2 = \delta_2$ and $x, y \geq 1$ then we have $|g(x) - g(y)| < \epsilon$.

Now, given $\epsilon > 0$ we set $\delta = \min\{\delta_1, \delta_2, \frac{1}{10}\}$. Then, if $|x-y| < \delta$ and $x, y > 0$ then at least one of $x, y \in (0,2]$ or $x, y \geq 1$ holds. In the first case $x, y \in (0,2] \subset [0,2]$, the condition $|x-y| < \delta \leq \delta_1$ implies $|f(x) - f(y)| = |g(x) - g(y)| < \epsilon$. In the second case $x, y \geq 1$, the condition $|x-y| < \delta \leq \delta_2$ implies $|f(x) - f(y)| = |g(x) - g(y)| < \epsilon$. Hence, $f(x)$ is uniformly continuous on $(0,+\infty)$.

□

Problem 9. $f(x) = x^2 \sin(1/x)$ *is uniformly continuous on* $[1, +\infty)$.

Sorry. This problem is not well-designed and too difficult.

Proof. Compute

$$f'(x) = 2x \sin(1/x) - \cos(1/x) = \frac{2\sin(1/x)}{1/x} - \cos(1/x).$$

Thus,

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{2\sin(1/x)}{1/x} - \cos(1/x) = \lim_{t\to 0^+} \frac{2\sin t}{t} - \cos t = 2 - 1 = 1.$$

Also, we have $|f'(1)| = |2\sin 1 - \cos 1| \leq 3$. Since $\lim_{x\to\infty} f'(x) = 1$, there exists $M > 1$ such that $|f'(x) - 1| < 1$ if $x > M$. Namely, $|f'(x)| < |f'(x) - 1| + 1 < 2$ if $x > M$.

Since $|f'(x)|$ is continuous for $x > 0$, $|f'(x)|$ attains its maximum N on the compact set $[1, M]$. Then, $N = \max_{1\leq x\leq M} |f'(x)| \geq |f'(1)| = 3 > 2 \geq |f'(y)|$ for $y > M$. So, N is the maximum of $|f'(x)|$ on $[1, +\infty)$.

Thus, given $\epsilon > 0$ if $|x - y| < \frac{\epsilon}{N}$ and $y \geq x \geq 1$ then by the fundamental theorem of calculus

$$|f(x) - f(y)| = |\int_x^y f'(t)dt| \leq \int_x^y |f'(t)|dt \leq \int_x^y N dt < \epsilon,$$

holds. Therefore, $f(x)$ is uniformly continuous on $[1, +\infty)$. ☐

Problem 10. *Assume that* $g(x)$ *is continuous on* $\mathbb{R}$ *and* $g(0) = 1$. *Then,* $f(x) = g(x)\cos(1/x)$ *is not uniformly continuous on* $(0, 1]$.

Hint: Theorem 23. (This theorem was an assignment.)

Proof. Since $g(x)$ is continuous on $\mathbb{R}$ and $g(0) = 1$, there exists some $\delta > 0$ such that $|g(x) - 1| < \frac{1}{10}$ for $|x| < \delta$.

We define $x_n = 1/(n\pi)$. Then, $\{x_n\}$ is a Cauchy sequence. Assume that $f(x)$ is uniformly continuous on $(0, 1]$. Then, $\{f(x_n)\}$ is also a Cauchy sequence. However, if $\frac{1}{n\pi} < \delta$ then the following hold

$$f(x_{2n}) = g(x_{2n})\cos(2n\pi) = g(x_{2n}) \geq \frac{9}{10},$$
$$f(x_{2n+1}) = g(x_{2n+1})\cos((2n+1)\pi) = -g(x_{2n+1}) \leq -\frac{9}{10}.$$

Namely, $f(x_{2n}) - f(x_{2n+1}) \geq \frac{9}{5}$. Contradiction. ☐

3. Improper integral

Problem 11. *Test the improper integral for convergence or divergence.*

$$\int_0^\infty \frac{x}{\sqrt{1+x^4}}dx.$$

Proof. We define $g(x) = \frac{x}{\sqrt{x^4}} = \frac{1}{x}$. Then,

$$\frac{f(x)}{g(x)} = \frac{x^2}{\sqrt{1+x^4}} = \frac{1}{\sqrt{x^{-4}+1}} \to 1$$

as $x \to \infty$. In addition, $\int_1^\infty \frac{1}{x}dx$ diverges. Hence, by the asymptotic comparison test, $\int_1^\infty f(x)$ also diverges. So, $\int_0^\infty f(x)$ diverges. □

Problem 12. *Test the improper integral for convergence or divergence.*

$$\int_e^\infty \frac{x}{(\ln x)^2\sqrt{1+x^4}}dx.$$

Proof. We define $g(x) = \frac{x}{(\ln x)^2\sqrt{x^4}} = \frac{1}{x(\ln x)^2}$. Then,

$$\frac{f(x)}{g(x)} = \frac{x^2}{\sqrt{1+x^4}} = \frac{1}{\sqrt{x^{-4}+1}} \to 1$$

as $x \to \infty$. In addition,

$$\begin{aligned}\lim_{t\to\infty}\int_e^t g(x)dx &= \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)}dx \\ &= \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^2}du = \lim_{t\to\infty} -\frac{1}{\ln t}+1 = 1.\end{aligned}$$

Hence, by the asymptotic comparison test, $\int_e^\infty f(x)$ also converges. □

Problem 13. *Assume that $\int_a^b f^2(x)dx$ and $\int_a^b g^2(x)dx$ converges. Prove that the improper integral converges.*

$$\int_a^b f(x)g(x)dx.$$

Proof. Since $\int_a^b \frac{1}{2}(f^2(x)+g^2(x))dx$ converges, by $|f(x)g(x)| \le \frac{1}{2}(f^2(x)+g^2(x))$ and the comparison test, $\int_a^b |f(x)g(x)|dx$ converges. Namely, $\int_a^b f(x)g(x)dx$ absolutely converges, and so converges. □

Problem 14. *Assume that $\int_0^1 f^2(x)dx$ converges. Prove that the improper integral converges.*

$$\int_0^1 f(x)x^{-\frac{1}{4}}dx.$$

Hint: Use the result of the previous problem.

Proof. We set $g(x) = x^{-\frac{1}{4}}$. Then, $\int_0^1 g^2(x)dx = \int_0^1 \frac{1}{\sqrt{x}}dx$ converges. Hence, by the proof of the previous problem, $\int_0^1 f(x)x^{-\frac{1}{4}}$ converges. □

4. Uniform convergence

Prove the following statements in this section.

Problem 15. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2(n+1)}$ *is continuous on* $\mathbb{R}$.

Proof. We set $u_n(x) = \frac{\sin nx}{n^2(n+1)}$. Then, $|u_n(x)| \le \frac{1}{n^2(n+1)} \le \frac{1}{n^3}$ and thus $\sum u_n$ uniformly converges by the theorem 22.2. Since u_n are continuous, $f(x)$ is continuous by the theorem 22.3. □

Problem 16. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2(n+1)}$ *is uniformly continuous on* $\mathbb{R}$.

Hint: First, show that $|f'(x)|$ is bounded.

Proof. We set $u_n(x) = \frac{\sin nx}{n^2(n+1)}$. Then, $u_n'(x) = \frac{\cos nx}{n(n+1)}$. Hence, $|u_n'(x)| \le \frac{1}{n(n+1)}$. Thus, for each x, we have

$$\sum_{n=1}^{m} |u_n'(x)| \le \sum_{n=1}^{m} \frac{1}{n(n+1)} \le \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = M,$$

for some constant M. Actually,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = (1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots = 1.$$

Therefore, $\sum u_n'$ uniformly converges by the theorem 22.2. Since u_n' are continuous, by the theorem 22.5, $f(x)$ is differentiable and $\sum u_n'$ converges to $f'(x)$.

Next, for each fixed x, we checked $\sum |u_n'(x)| \le M$. Hence, $-|u_n'(x)| \le u_n'(x) \le |u_n'(x)|$ yields

$$-M \le -\sum_{n=1}^{m} |u_n'(x)| \le \sum_{n=1}^{m} u_n'(x) \le \sum_{n=1}^{m} |u_n'(x)| \le M.$$

Thus, the limit location theorem shows $|f'(x)| \le M (= 1)$. Thus, given $\epsilon > 0$ if $|x - y| \le \frac{\epsilon}{M}$ with $y \ge x$ then

$$|f(x) - f(y)| = |\int_x^y f'(t)dt| \le \int_x^y M dy < \epsilon.$$

namely $f(x)$ is uniformly continuous. □

Problem 17. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2(x^2+1)}$ *attains its maximum on* $\mathbb{R}$.

Hint: First, show that $\displaystyle\lim_{|x|\to+\infty} f(x) = 0$.

Proof. We set $u_n(x) = \frac{\sin nx}{n^2(x^2+1)}$. Then, $|u_n(x)| \leq \frac{1}{n^2}$ and thus $\sum u_n$ uniformly converges by the theorem 22.2. Since u_n are continuous, $f(x)$ is continuous by the theorem 22.3.

Next, we define $\sum \frac{1}{n^2} = L$. Then,

$$|\sum_{k=1}^{n} u_k(x)| \leq \sum_{k=1}^{n} |u_k(x)| = \sum_{k=1}^{n} \frac{|\sin kx|}{k^2(x^2+1)} \leq \frac{1}{x^2+1} \sum_{k=1}^{n} \frac{1}{k^2} \leq \frac{L}{x^2+1}.$$

Hence, passing n to $+\infty$ yields

$$|f(x)| \leq L(x^2+1)^{-1}. \tag{1}$$

which means $f(x) \to 0$ as $|x| \to +\infty$.

Now, we can observe $f(0) = \sum u_n(0) = \sum 0 = 0$. Thus, if $f(x) \leq 0$ for all $x \in \mathbb{R}$ then $\max_{\mathbb{R}} f(x) = 0$. Hence, we may assume that there exists a point $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$. Then, by (1) there exists $M > 0$ such that $|f(x)| < f(x_0)$ for all $x > M$. On the other hand, the continuous function $f(x)$ attains its maximum on the compact interval $[-M, M]$. Namely, there exists $x_1 \in [-M, M]$ such that $f(x) \leq f(x_1)$ for all $x \in [-M, M]$. Then, $f(x_1) \geq f(x_0) > |f(x)|$ for $|x| > M$ implies that $f(x_1)$ is the maximum on $f(x)$ on $\mathbb{R}$. □

5. Analysis in $\mathbb{R}^2$

Problem 18. *Assume that a continuous function $f(x,y)$ satisfies*

$$f(r,q) = r - q$$

for all $r, q \in \mathbb{Q}$. Prove that $f(x,y) = x - y$ holds for all $(x,y) \in \mathbb{R}^2$.

Proof. Given $x \in \mathbb{R}$, we can choose a sequence of rational numbers $\{r_n\}$ converging to x by the theorem 25. Then, given $q \in \mathbb{Q}$, we have $\lim(r_n, q) = (x, q)$. Thus, the theorem 24.5B yields

$$x - q = \lim(r_n - q) = \lim f(r_n, q) = f(x, q).$$

Next, given $y \in \mathbb{R}$, we choose a sequence of rational numbers $\{q_n\}$ converging to y. Thus,

$$x - y = \lim(x - q_n) = \lim f(x, q_n) = f(x, y).$$

□

Problem 19. *Assume that a positive continuous function $f(x,y)$ satisfies $\lim\limits_{\|(x,y)\| \to +\infty} f(x,y) = 0$. Prove that $f(x,y)$ attains its maximum, while it does not have its minimum.*

Proof. Let us denote $f(0,0)$ by c. Then, $c > 0 = \lim_{\|(x,y)\|\to+\infty} f(x,y)$ implies that there exists some constant $R > 0$ such that $f(x,y) < c$ holds if $\|(x,y)\| > R$.

Next, we define a bounded set $K = \{(x,y) : \|(x,y)\| \leq R\}$. Since $\|(x,y)\| = \sqrt{x^2+y^2}$ is a continuous function on $\mathbb{R}^2$, the set K is closed by the theorem 25.1B. Hence, the theorem 25.2 implies that K is a compact set. Hence, by the theorem 24.7B, the continuous function $f(x,y)$ attains its maximum $f(\bar{x},\bar{y})$ on K. In addition, $f(\bar{x},\bar{y}) \geq f(0,0) > f(x,y)$ for $(x,y) \notin K$. Namely, $f(\bar{x},\bar{y})$ is the maximum of f on $\mathbb{R}^2$.

Assume that f attains its minimum $f(\tilde{x},\tilde{y})$. Since

$$f(\tilde{x},\tilde{y}) > 0 = \lim_{\|(x,y)\|\to+\infty} f(x,y),$$

there exists a point $(\tilde{x}',\tilde{y}')$ such that $f(\tilde{x},\tilde{y}) > f(\tilde{x}',\tilde{y}')$. Contradiction. $\square$

Problem 20. $f(x,y) = \dfrac{xy}{x^2+y^2+1}$ *is uniformly continuous on* $\mathbb{R}^2$.

Note: It would be a bit challenging to prove the statement above.

Proof. Given $x \in \mathbb{R}$, we define a differentiable function $g_x(y) = \frac{xy}{x^2+y^2+1}$. Then,

$$\begin{aligned}|g_x'(y)| =& \left|\frac{x}{x^2+y^2+1} - \frac{2xy^2}{(x^2+y^2+1)^2}\right| = \left|\frac{x(x^2-y^2+1)}{(x^2+y^2+1)^2}\right| \\ \leq& \frac{|x|}{x^2+y^2+1} \leq \frac{|x|}{x^2+1} \leq \frac{1}{2}.\end{aligned}$$

Similarly given $y \in \mathbb{R}$, we define a differentiable function $h_y(x) = \frac{xy}{x^2+y^2+1}$. Then, $|h_y'(x)| \leq \frac{1}{2}$.

Given $\epsilon > 0$, if $\|(x_1,y_1) - (x_2,y_2)\| < \epsilon$ then we have

$$\epsilon > \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \geq \max\{|x_1-x_2|, |y_1-y_2|\}.$$

Then, the FTC yields

$$|f(x_1,y_1) - f(x_2,y_1)| = |h_{y_1}(x_1) - h_{y_1}(x_2)| = \left|\int_{x_1}^{x_2} h_{y_1}'(t)dt\right|.$$

Hence,

$$\begin{aligned}|f(x_1,y_1) - f(x_2,y_1)| \leq& \int_{\min\{x_1,x_2\}}^{\max\{x_1,x_2\}} |h_{y_1}'(t)|dt \\ \leq& \int_{\min\{x_1,x_2\}}^{\max\{x_1,x_2\}} \frac{1}{2}dt = \frac{|x_1-x_2|}{2} < \frac{\epsilon}{2}.\end{aligned}$$

In the same manner, we have $|f(x_2,y_1) - f(x_2,y_2)| < \frac{\epsilon}{2}$. Hence,

$$|f(x_1,y_1) - f(x_2,y_2)| \leq |f(x_1,y_1) - f(x_2,y_1)| + |f(x_2,y_1) - f(x_2,y_2)| < \epsilon.$$

Therefore, $f(x, y)$ is uniformly continuous. $\square$

Theorem 21. *Assume that $f(x, y)$ is uniformly continuous on $S \subset \mathbb{R}^2$, and $\{(a_n, b_n)\}$ is a Cauchy sequence in S. Then, $\{f(a_n, b_n)\}$ is a Cauchy sequence.*

Proof. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$ if $\|(x_1, y_1) - (x_2, y_2)\| < \delta$. Since $\{(a_n, b_n)\}$ is a Cauchy sequences, there exists N such that $\|(a_n, b_n) - (a_m, b_m)\| < \delta$ if $n, m \geq N$. Hence,

$$|f(a_n, b_n) - f(a_m, b_m)| < \epsilon$$

holds for $n, m \geq N$. Thus, $\{f(a_n, b_n)\}$ is a Cauchy sequence. $\square$

Theorem 22. *Let $S = \{(x, y) : x^2 + 4y^2 \leq 9, x \geq 1\}$ and $f(x, y)$ is continuous on S. Show that f is bounded on S.*

Proof. First of all, if $(x, y) \in S$ then $\|(x, y)\| = \sqrt{x^2 + y^2} \leq \sqrt{x^2 + 4y^2} \leq \sqrt{9} = 3$. Namely, S is bounded.

Next, we consider the continuous functions $g(x, y) = x^2+4y^2$ and $h(x, y) = x$. Then, the theorem 25.1B implies that $G = \{(x, y) : x^2 + 4y^2 \leq 9$ and $H = \{(x, y) : x \geq 1\}$ are closed. Hence, by the theorem 25.1A $S = G \cap H$ is closed. Thus, by the the theorem 25.2 and the theorem 24.7A S is compact and thus f is bonded on S. $\square$

6. Theorems

The following theorems are not given in the textbook, but you can cite during the final.

Theorem 23 (Pset 5, problem 5-(a))**.** *Assume $f(x)$ is uniformly continuous on I, and $\{a_n\}$ is a Cauchy sequence in I. Then, $\{f(a_n)\}$ is a Cauchy sequence.*

Theorem 24. $\sin x$ *and* $\cos x$ *are continuous on* $\mathbb{R}$. *Moreover, they has continuous derivatives* $(\sin x)' = \cos x$ *and* $(\cos x)' = -\sin x$, *respectively.*

Theorem 25. *Given two real numbers $x < y$, there exists a rational number r such that $x < r < y$. Moreover, given real number x, there exists a sequence $\{r_n\}$ of rational numbers such that* $\lim r_n = x$.